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NON-GAUSSIAN PRICE DYNAMICS

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ABSTRACT

It is well known that the probability distribution of stock returns is non-gaussian. The tails of the distribution are too "fat", meaning that extreme price movements, such as stock market crashes, occur more often than predicted given a gaussian model. Numerous studies have attempted to characterize and explain the fat-tailed property of returns. This is because understanding the probability of extreme price movements is important for risk management and option pricing. In spite of this work, there is still no accepted theoretical explanation. In this chapter, we use a large collection of data from three different stock markets to show that slow fluctuations in the volatility, (i.e., the size of return increments) coupled with a gaussian random process, produce the non-gaussian and stable shape of the return distribution. Furthermore, because the statistical features of volatility are similar across stocks, we show that their return distributions collapse onto one universal curve. Volatility fluctuations influence the pricing of derivative instruments, and we discuss the implications of our findings for the pricing of options.

INTRODUCTION

In his thesis, *Théorie de la Spéculation* (1900), Louis Bachelier modelled price differences as a simple random process (Bachelier, 1900; 1964). It was a seminal publication, not only was it the first mathematical study of stock prices, it also was the first time that the diffusion of a markovian process was treated analytically, pre-dating by five years the work of Albert Einstein (1905). Bachelier's work layed the foundation for the field of mathematical finance; a field that has blossomed in the last century.

Although pioneering for its time, several modifications to Bachelier's random walk model have been needed. First was the realization that prices move in relative amounts rather than absolute amounts, and that *returns* rather than price differences should be modelled as a random process (Osborne, 1959). Next, several papers showed that returns could not be described by a simple random process because extreme price movements occur much to frequently, causing the return distribution to have *fat* tails (Mandelbrot, 1963; Fama, 1965).

Despite numerous attempts to explain the fat-tailed nature of the return distribution (Mantegna and Stanley, 1995; Bouchaud and Potters, 2003; Gabaix et Al. 2003; Viswanathan et Al., 2003; Farmer et Al. 2004; Bassler et Al. 2007), there is still no consensus on the underlying cause.

Characterizing the shape and scaling of the return distribution is important because it determines the probability of observing extreme events, which is needed for proper risk management and for the correct pricing of derivative instruments. If returns are gaussian, then options are priced according to the standard equation of Black and Scholes (1973). Because returns are ill-described by a gaussian process, B-S prices exhibit systematic biases across moneyness and time to maturity.

Two explanations for the non-gaussian shape of the return distribution are often discussed in the literature. The first is known as the *mixture-of-distributions* hypothesis (MOD) (Praetz 1972; Blattberg and Gonedes, 1974), which states that the return distribution is a mixture of gaussian distributions with different

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variances. The second is known as the *stable Paretian* hypothesis (SP) (Mandelbrot, 1963; Mantegna and Stanley, 1995; Lux 1996), which states that returns are pulled independently and identically from a stable or truncated stable distribution. The MOD hypothesis better describes the shape of the distribution (Blattberg and Gonedes, 1974), but the SP hypothesis better describes the stability of the distribution, i.e., that the distribution retains it's non-gaussian shape when aggregating returns over longer timescales.

In this chapter, we present a model that reproduces both the shape and stability of the return distribution. The model is based on two assumptions about the characteristics of *volatility*, the scale of returns. First, we assume that volatility fluctuates slowly, i.e., volatility fluctuations are rather small over short to intermediate intervals (days to weeks), but are quite large over longer timescales (months to years). Second, we assume that the process that generates volatility fluctuations is such that the inverse square of volatility is gamma-distributed. As shown below, these assumptions predict that the return distribution will be non-gaussian (specifically Student *t*-distributed), and that the return distribution will keep this shape for short to intermediate timescales. The model, therefore, reproduces empirical results that previously seemed contradictory, and that individually, were used to support one or the other of the two competing hypotheses for non-gaussian returns (MOD or SP).

We test the predictions of the model using data from 6 stocks collected from 3 global exchanges over different time periods. The model performs well for each stock, suggesting it is robust to different time periods, different market sectors, and different countries. These results have implications for the pricing of options, which we discuss after presenting the results.

THEORETICAL APPROACH

We define the return at time t as the difference in logarithmic price from time t to time $t+\tau$,

$$r(t) = \ln[p(t+\tau)] - \ln[p(t)], \tag{1}$$

where the price, *p(t)*, is defined as the midpoint price between the best bid price and offer price in the market (these prices are known as *quotes*).

There are several possibilities to set the unit of the time index, *t*, and here we study returns over the finest possible time scale, *event-time*. In event-time, *t* is updated, incremented by a unit, whenever there is a change in the midpoint between the prevailing best quotes.

To begin our analysis we will use the Langevin approach. The proposed stochastic dynamic for the return *during a day* will be the fundamental ordinary differential equation

$$\frac{dr}{dt} = \sigma\xi(t), \tag{2}$$

Where $\xi(t)$ is a white gaussian noise of unit variance, i.e.,

$$\langle \xi(t) \rangle = 0, \tag{3}$$

$$\langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{4}$$

t is the time, and σ is the *volatility*, or the strength of the noise acting on the return.

Under Ito's calculus, this stochastic dynamic leads immediately to an evolution equation for the probability of the returns, the Fokker-Planck equation, which is

$$\frac{\partial P(x,t)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 P(x,t)}{\partial x^2} \ . \tag{5}$$

This equation has an analytical solution, which is the same that Bachelier found for non *log*-transformed prices in 1900, and five years after him Einstein suggested as the distribution for Brownian particles: the normal distribution. “The problem, which corresponds to the diffusion from a single point (ignoring the interactions between diffusing particles) is now mathematically completely defined: its solution is”

$$P(r,t) = \sqrt{\frac{1}{2\pi\sigma^2 t}} \exp\left(- \frac{r^2}{2\sigma^2 t} \right), \tag{6}$$

"Therefore, the distribution of the resulting displacements in a given time *t* is the same as random error...", from (Einstein, 1905), author's translations.

In Eqs. (2-6), the volatility is fixed, meaning that during any given day, we assume a constant strength for the noise. At this point, the model is nearly identical to Bachelier's original work (using returns instead of price differences). To generate the non-gaussian, stable shape of the return distribution, we assume that volatility slowly fluctuates. Specifically, we assume that volatility is sufficiently slow varying, such that we can treat σ as a constant over intraday time scales. Over longer periods, we assume that the square of the volatility (the local variance of the price), $v=\sigma^2$, follows a feedback process (Bouchaud and Potters, 2003)

$$v_{k-1} - v_0 = (1-\varepsilon)(v_k - v_0) - g\varepsilon v_k (\langle \varsigma \rangle - \varsigma) \ , \tag{7}$$

where ς is a centered noise term. This process produces a local variance that is mean-reverting to the value v_0, but which retains memory of past values with coupling parameter *g*.

The continuous formulation of this discrete Langevin equation leads to the following Fokker-Plank equation for the evolution of the probability

$$\frac{\partial P(v,t)}{\partial t} = \varepsilon \frac{\partial \ (v_k - v_0) P(v,t)}{\partial v} + Dg^2 \varepsilon^2 \frac{\partial^2 P(v,t)}{\partial v^2} \ , \tag{8}$$

where *D* is the variance of the noise ς. The stationary solution for this equation is an inverse-gamma distribution

$$f(v) = \frac{1}{\Gamma(\mu) v^{1+\mu}} \exp(-v_0 / v) \ , \tag{9}$$

where μ=1+ $Dg^2\varepsilon$. Now, if we define

$$v = \frac{1}{\beta}, \tag{10}$$

$$\mu = \frac{n}{2}, \tag{11}$$

$$\sigma_0 = \frac{n}{2\beta_0}, \tag{12}$$

so, that the return distribution can be written as

$$P(r,t\,|\,\beta) = \sqrt{\frac{\beta}{2\pi\tau}}\exp\frac{\beta r^2}{2\tau}\ , \tag{13}$$

and using the distribution's transformation

$$P(\beta) = P(\sigma^2)\left|\frac{d\sigma^2}{d\beta}\right|\ , \tag{14}$$

we can write

$$f(\beta) = \frac{1}{\Gamma(n/2)}\left(\frac{n}{2\beta_0}\right)^{n/2}\exp\left(-\frac{n\beta}{2\beta_0}\right), \tag{15}$$

which is the equation for a gamma distribution. On any single day, the distribution of returns is gaussian with variance $v = \sigma^2 = 1/\beta$, as shown in Eq. (6). Because β can vary at longer time scales, the return distribution observed with data pulled from many different days is obtained by marginalizing over β,

$$P(r,\tau) = \int P(r,\tau\,|\,\beta)f(\beta)d\beta\ . \tag{16}$$

As seen in this equation, the return distribution is a mixture of Gaussians with different variances.

A straightforward integration of the conditional probability of returns, $P(r,\tau\,|\,\beta)$, and the distribution $f(\beta)$ yields the following for the return distribution

$$P(r,\tau) = \frac{\Gamma\left[(n+1)/2\right]}{\Gamma[n/2]}\sqrt{\frac{\beta_0}{2\pi\tau}}\left(1+\frac{\beta_0 r^2}{n\tau}\right)^{-(n+1)/2}\ , \tag{17}$$

which is a variant of the Student's *t*-distribution. The non-Gaussian shape of the distribution results from collecting returns from time periods separated by long intervals where β is different. The stability of this shape for short to intermediate τ results from negligible fluctuations of β over these time scales.

Other papers have reported that returns follow a Student's *t*-distribution and have fit returns to a generic version of this distribution, see (Praetz, 1972; Blattberg and Gonedes, 1974; Bouchaud and Potters, 2003) for examples. Eq. (17) does not represent a fit to the return data, but is determined solely by the two parameters, β_0 and *n*, from the distribution of the inverse variance, β. In the results we present below, we do not fit a Student's *t*-distribution, but instead compare the empirical distribution to the predicted distribution as expressed in Eq. (17) and as determined by the independent measurement of β_0 and *n*. This specifically tests our model rather than the more general result that returns follow a Student's *t*-distribution.

To facilitate the presentation of the empirical results, we define the following normalized variables

$$r' = r\sqrt{\frac{2\beta_0}{n\tau}\frac{1}{\beta}}, \qquad (18)$$

$$P'(r') = \left[\Omega P(r,\tau)\right]^{2/(n+1)}, \qquad (19)$$

where $\Omega = \sqrt{2\pi}\Gamma[n/2]/\Gamma[(n+1)/2]$. These normalizations allow results for different time scales and different stocks to collapse on a single curve.

DATA

Our results are produced using a large amount of data (of the order of 10^7 data points) from three stock markets over three time periods: the London Stock Exchange (LSE) from May 2, 2000 to December 31, 2002, the New York Stock Exchange (NYSE) from January 2, 2001 to December 31, 2002, and the Spanish Stock Exchange (SSE) from January 2, 2004 to December 29, 2006. For each market, we choose two highly traded stocks that are from different market sectors. From the NYSE we study General Motors (GM), an automotive maker, and International Business Machines (IBM), a computer hardware/software maker and consulting firm. From the LSE we study AstraZeneca (AZN), a pharmaceutical company, and Vodafone (VOD), a mobile telecommunications company. From the SSE we study Santander (SAN), a banking group, and Telefonica (TEF) a broadband and telecommunications company. We consider only the electronic markets for these stocks, and we measure returns whenever the mid-price of a stock fluctuates. This approach allows us to study returns on the finest possible time scale. When aggregating returns over longer time scales, we use non-overlapping intervals. As mencioned, we measure price fluctuations, or returns, in the standard way (Bouchaud and Potters, 2003) as $r(t) = \ln[p(t+\tau)] - \ln[p(t)]$, where p is the mid-price, t is the time (which is updated by one unit whenever the mid-price changes), and τ is the time increment (also measured in units of mid-price changes).

RESULTS

In this section, we compare empirical results with the assumptions and results of the model. We present supporting evidence for our assumption that the inverse variance of stock returns is gamma distributed. In addition, we show that the return distribution collapses over intraday time scales, supporting our assumption of a slowly fluctuating volatility. Finally, we plot the scaled return distributions for all of the stocks in our study; the collapse of these distributions suggests that the volatility characteristics we've assumed are universally valid.

Figs. 1 and 2 present results only for the stock IBM, although not shown, the results for the other stock in our study are similar in appearance. In Fig. 1(a) we plot the probability density function of β. We overlay the plot with the best fit gamma distribution, i.e., the gamma distribution using maximum likelihood estimates (MLEs) for the parameters n and β_0. These MLEs for IBM and the other stocks are reported in Table 1. In Fig. 1(b), we plot the complementary cumulative distribution (CCD) of β and again overlay the plot with the best fit gamma distribution (the CCD is the integral of the probability function). As seen in both plots, the gamma distribution fits well.

In Fig. 2(a) we plot the probability density function for IBM scaled returns, *r'*, from $\tau = 10$ to $\tau = 640$, which is up to one trading day for the stocks in our study. From the MLEs for parameters n and β_0, we predict the full probability distribution of returns, as derived in Eq. (17) and overlay this prediction on the plot. In Fig. 2(b) we plot the CCD of absolute scaled returns, $C(|r'|)$. We show this plot in logarithmic coordinates to focus on the tails of the distribution, and we overlay the plot with the CCD of the theoretical distribution. As seen in both plots, the model matches the data well in the central region and the tails, and the shape of the distribution is stable over these time scales.

In Fig. 3, we plot the empirical CCD of β versus the fit for all six stocks in the study. This plot is created by first fixing the value of the fit *C(.)*, calculating β at this point, and then plotting the value of the empirical *C(.)* for this β. The plot is sometimes called a P-P plot, which is used to assess the similarity of the distributions on the *x* and *y* axes. If the empirical distribution follows the fitted distribution exactly, the curve will lie on the 45º line. The empirical data shows no systematic deviations from a gamma distribution.

Our model predicts that the functional form of the return distribution is the same for different stocks, and that inconsistencies can be attributed to different parameters of the gamma distribution for β. This is verified in Fig. 4, where we show the collapse of the renormalized probability distribution *P'(r')*, Eq. (19), for all 6 stocks in our study. The return distributions are well fit by the model and collapse over the entire range of returns.

Security	Events	Events/Day	n	$\beta_0 (\times 10^6)$
GM	505541	1025	3.7	6.3
IBM	1056636	2143	4.4	12.8
AZN	1013482	1501	4.8	5.0
VOD	841492	1247	8.8	1.8
SAN	243545	329	15.7	3.0
TEF	315093	426	30.6	4.8

Table 1. The total number of price changing events, average number of events per day, and maximum likelihood estimates for n and β_0 for the six stocks studied.

IMPLICATIONS FOR OPTION PRICING

In the option pricing paper of Black and Scholes (1973), stock prices are assumed to follow geometric brownian motion with a constant volatility. Because volatility is constant, there exists only one source of randomness that affects both the price of the stock and the price of an option on the stock. The main insight of the B-S model is the following: by holding a certain proportion of stock together with the option (which is dynamically rebalanced), the randomness of both can be cancelled so that a riskless portfolio results. By setting the return of this portfolio to the risk-free rate, the price of the option can be deduced.

The first study to price options with nonconstant volatility was Merton (1976), who allowed the price process to include random but diversifiable jumps. A number of papers followed that allow for more general changes in volatility. These papers can be separated into two groups, the first group keeps only one source of randomness and assumes that volatility is a deterministic function of price and time (Cox and Ross, 1976; Derman and Kani, 1994; Platen, 2001). The models in these papers are known as *local volatility models*. The second group allows for volatility to be driven by a second source of randomness (Hull and White, 1987; Stein and Stein, 1991; Heston, 1993). These models are known as *stochastic volatility models*. Option pricing within local volatility models can be solved in a similar way as in the B-S model, by forming riskless portfolios. Within stochastic volatility models, pricing options requires either the introduction of a hedging instrument for volatility fluctuations, or assumptions about the risk preferences of investors.

As seen in the figures above, the returns for the stocks in our study are Student *t*-distributed and volatility exhibits slow dynamics. The B-S model no longer applies under these circumstances. Calculating options prices when volatility exhibits both extreme fluctuations and slow dynamics is non-trivial

Local volatility models that produce student *t*-distributed returns include the models found in Borland (2002) and the minimal market model of Platen (2001). Stochastic volatility models that produce student *t*-distributed returns include the GARCH model (Nelson, 1990) and the 3/2 model (Lewis, 2000). The stochastic volatility model that we adopt in this chapter is a variation of the GARCH model. Option pricing within the GARCH model was treated in Satchell and Timmermann (1992), Amin and Ng (1993), and Duan (1995). Care should be taken when interpreting these results because volatility fluctuations are too fast over longer time-periods within the GARCH model. Taylor (2000) studies option pricing when volatility is slowly varying.

To obtain correct option prices, the full dynamics of volatility from short to long timescales must be specified accurately. There is currently no consensus on a volatility model that acheives such a specification, and many of the volatility models above are able to reproduce in a general way the biases observed in B-S option pricing.

CONCLUSIONS

In this chapter we have presented evidence that the non-gaussian shape and stable scaling of the return distribution are due to slow, but significant, fluctuations in volatility. Furthermore, our results suggest that return distributions for stocks from different exchanges, time periods, and over different time scales can be described by one functional form.

Our model decomposes individual returns into the product of two terms: a gaussian noise term and a volatility parameter. On any single day, we assume that volatility is constant so that returns are well described by gaussian fluctuations. Across many days, however, volatility is driven by a mean-reverting process that produces a gamma distributed β. When combining the local gaussian behavior of returns with these slow volatility fluctuations, the result is a Student's *t*-distribution for returns that appears stable over short to intermediate time scales.

The idea that volatility fluctuations produce non-gaussian returns is not new. It was originally suggested several decades ago (Praetz 1972; Blattberg and Gonedes, 1974; Clark 1973). This has competed with an alternative explanation that returns are drawn unconditionally from a fat-tailed, stable distribution (Mandelbrot 1963; Mantegna and Stanley, 1995; Lux 1996). Our model can reproduce both the non-gaussian shape and the apparent stability of the return distribution, two characteristics that previously seemed to be at odds with one another and that individually could be used to support one or the other competing explanations.

Using intraday data for 6 stocks from 3 countries, we confirm the predictions of the model. We find that the inverse square of daily volatility is well fit by a gamma distribution. Using the parameters from this fit, we compute the return distribution from the model and find that the empirical distribution matches this prediction extremely well. Furthermore, we find that the return distribution collapses over intraday timescales, a result that supports our assumption of constant intraday volatility. Finally we show that, by appropriately rescaling the returns for each of the stocks in our study, their return distributions collapse onto a single curve, confirming that our model is valid for a variety of different stocks.

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