The Hochschild and Lie (co)homology of the algebra of pseudodifferential operators in one and several variables

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Abstract

We describe the Hochschild (co)homology groups of the associative algebra Ψ_n of pseudodifferential symbols in $n \geq 1$ independent variables. We prove in particular that the first Hochschild (co)homology group $HH^1(\Psi_n)$ is $2n$- dimensional. Also, we give an elementary calculation of the first Lie (co)homology group $H^1_{Lie}(\Psi_n)$ of Ψ_n equipped with the Lie bracket induced by its associative algebra structure.

1 Introduction

The algebra of pseudodifferential symbols in one variable is a standard object in the theory of integrable systems, see for instance [Di, KW]. In the case of pseudodifferential symbols in several variables there is a far less extensive literature. We mention the papers

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by Dzhumadil´daev [D], and Parshin [P] and the classical work by Wodzicki [Wod]. It is known that in the case of one variable there is exactly one central extension for the *Lie algebra* of pseudodifferential symbols, see [F] and [KW], and it has been also observed that this central extension is of interest for integrable systems, see for instance the classical note [RS] and treatise [KW]. In this paper we investigate the existence and classification of exterior derivations for the case of n independent variables, that is, we compute $HH^1(\Psi_n)$, and then we consider the classification of central extensions of Ψ_n as a Lie algebra. Our calculations (in the Lie algebra case) provide an essential simplification of the work [D] on the Lie algebra cohomology of Ψ_n. Thus, our work can be useful for the construction of integrable systems in an arbitrary number of independent variables. We believe that this is interesting because it complements known constructions of integrable systems of equations in several independent variables such as the KadomtsevPetviashvili hierachy and their relatives, [Di], the examples of K. Tenenblat and her coworkers [T], and the examples in Parshin's paper [P]. In this paper we consider only the algebraic classification problem. Explicit applications to mathematical physics appear in the companion paper [BR].

2 Hochschild homology and cohomology of pseudodifferential operators

2.1 The objects

Let $\mathbb{K}$ be a field of characteristic zero. The Weyl algebra, or the algebra of algebraic differential operators in affine space can be described as the vector space $A_n = \mathbb{K}[\{x_{\pm i} : i = 1, \ldots, n\}]$, with the multiplication law determined by the rules

$$[x_{+i}, x_{+j}] = 0 = [x_{-i}, x_{-j}],\ [x_{-i}, x_{+j}] = \delta_{ij}\ .$$

This algebra acts faithfully on $k[x_1, \ldots, x_n]$ sending x_{+i} to the multiplication by x_i, and x_{-i} to $\frac{\partial}{\partial x_i}$. This algebra is filtered by the order of the differential operators, and it also has the so-called Berstein filtration, in which a monomial $x_{+1}^{a_1} \cdots x_{+n}^{a_n} x_{-1}^{b_1} \cdots x_{-n}^{b_n}$ has total degree $\sum_i a_i + \sum_i b_i$.

We may localize on the x_{+i}'s, or on the x_{-i}'s, but not on both simultaneously, unless we consider formal series on $x_{\pm i}^{-1}$.

Through this paper, we shall use the usual multi-index notation. Thus, given $I = (i_1, ..., i_n) \in \mathbb{Z}^n$, we set $x_+^I = x_{1+}^{i_1} \cdots x_{n+}^{i_n}$ and similarly $x_-^J = x_{1-}^{j_1} \cdots x_{n-}^{j_n}$. We write $J \leq M$ for $M = (m_1, ..., m_n) \in \mathbb{Z}^n$ if $j_i \leq m_i$ for each i, and write $J \geq 0$ for $J \geq (0, ..., 0)$. Then, the space of pseudodifferential operator in n variables will mean the following associative algebra

$$\Psi_n = \left\{ \sum_{I,J \in \mathbb{Z}^n} a_{IJ} x_+^I x_-^J : a_{IJ} \in \mathbb{K}, a_{IJ} = 0 \text{ if } I, J \leq M \text{ for some } M \in \mathbb{Z}^n \right\}$$

The multiplication rule may be described by the following formula

$$FG = \sum_{K\in\mathbb{Z}^n} \frac{1}{K!} : \partial_-^K(F)\partial_+^K(G) :$$

where $F, G \in \Psi_n$ and $\partial_{k+}(x_+^I) = i_k x_+^{I-\epsilon_k}$, $\partial_{k-}(x_-^J) = j_k x_+^{J-\epsilon_k}$, $\partial_\pm(x_\mp^M) = 0$, $\epsilon_k = (0, ..., 1, ..., 0) \in \mathbb{Z}^n$.

The notation $: \star :$ means normal ordering. For example, if $n = 1$

$$\partial_-(x_+x_-^j)\partial_+(x_+^i x_-) =: jx_+x_-^{j-1} i x_+^{i-1} x_- := ijx_+^i x_-^j$$

It is clear that $A_n \subset \Psi_n$ is a subalgebra. The algebra Ψ_n has also two filtrations: one by "order of differential operator", namely $|x_+^I x_-^J|_{diff} = |J| = \sum_i j_i$, and also by total degree: $|x_+^I x_-^J|_{tot} = |I| + |J|$. We call this total degree the Bernstein filtration. The associated graded algebra is commutative, we may identify it with Laurent polynomials $\mathrm{gr}\Psi_n = k[x_{+1}, x_{+1}^{-1}, \dots x_{+n}, x_{+n}^{-1}, x_{-1}, x_{-1}^{-1}, \dots x_{-n}, x_{-n}^{-1}]$. On the other hand, we note that the algebra Ψ_n is *complete* with respect to the Bernstein filtration.

2.2 Main tools

By a filtered abelian group M we mean a family of subgroups F_pM for each $p \in \mathbb{Z}$ such that $F_{p+1}M \subseteq F_pM$ for all p, $\cup_p F_pM = M$, and $0 = \cap_p F_pM$. In such situation, two notions of completions $\widehat{M}$ of M, can be considered

- (Algebraic completion) There is a canonical inverse system $\{M/F_{p+1}M \to M/F_pM\}_{p\in\mathbb{Z}}$ and $\widehat{M}$ can be defined using the inverse limit, satisfying certain universal property that characterize it, but that can also be described explicitly as a sub object of the product

  $$\widehat{M} := \lim_{\leftarrow p} M/F_p = \{(\overline{m}_p)_p \in \prod_{p\in Z} M/F_pM : \overline{m}_p \equiv \overline{m}_{p+1} \forall p\}$$

  There is a canonical map $M \to \widehat{M}$ given by $m \mapsto (\overline{m}_p)_p$ where $\overline{m}_p = m \bmod F_pM$, and this map is injective because $\cap_p F_pM = 0$.

- (Metric completion) Fix a real number $r > 1$ and define $||0|| = 0$, $||m|| = r^p$ if $m \in F_pM$ and $m \notin F_{p-1}M$. Notice that a sequence of elements (of M) $m_0, m_{-1}, m_{-2}, \dots$ with $m_{-i} \in F_{-i}M$ verifies $||m_{-n}|| \to 0$. This adic-norm verifies (a stronger version of) the triangular inequality

  $$||m + m'|| \leq max\{||m||, ||m'||\} \leq ||m|| + ||m'||$$

  and hence makes M an (ultra)metric space by declaring the distance between m and m' to be $||m - m'||$. Notice that the properties $\cup F_pM = M$ and $\cap F_pM = 0$ make $||-||$ a well-defined function and $||m|| = 0$ if and only if $m = 0$. One may define $\widehat{M}$ as the usual metric completion using classes of Cauchy sequences of M.

Despite the apparent different approaches, it is standard that both completions yield the same object (see for instance [L, Theorem 10.1]), the inclusion $M \to \widehat{M}$ of the first construction is nothing but viewing M as a dense subset of $\widehat{M}$.

On several arguments we will use the following standard Lemma of filtered abelian groups.

Lemma 2.1. *Let Z, A, B, C be filtered abelian groups and*

$$Z \xrightarrow{h} A \xrightarrow{f} B \xrightarrow{g} C$$

a complex of filtered groups (i.e. each morphism preserves the respective filtration). If the induced sequence

$$\mathrm{gr}Z \xrightarrow{\mathrm{gr}h} \mathrm{gr}A \xrightarrow{\mathrm{gr}f} \mathrm{gr}B \xrightarrow{\mathrm{gr}g} \mathrm{gr}C$$

is exact in A and B, then, the following holds:

1. *If the filtrations are bounded below, i.e. if $F_{p_0}M = 0$ for some $p_0 \in \mathbb{Z}$ ($M = A, B, C$), then the original complex*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact in B.

2. *In the general case, the completion*

$$\widehat{A} \xrightarrow{\widehat{f}} \widehat{B} \xrightarrow{\widehat{g}} \widehat{C}$$

is exact in B.

Recall that a chain map $f\colon (A, \partial_A) \to (B, \partial_B)$ between two complexes of abelian groups is called a quasi-isomorphism if the induced map $f_*\colon H_*(A) \to H_*(B)$ is an isomorphism. An interesting consequence of the lemma 2.1 is the completed version of Künneth formula:

Corollary 2.2. *Let A, B, C be completed filtered complexes such that $\mathrm{gr}(A\widehat{\otimes}B) = \mathrm{gr}A \otimes \mathrm{gr}B$ and let $f\colon C \to A\widehat{\otimes}B$ a map such that $\mathrm{gr}f : \mathrm{gr}C \to \mathrm{gr}A \otimes \mathrm{gr}B$ is a quasi-isomorphism. Then the original map is a quasi-isomorphism.*

Proof. Recall first the Cone construction. If $\phi : (X, d_X) \to (Y, d_Y)$ is a morphism of complexes (with $d(X_n) \subseteq X_{n+1}$), then the cone of f is defined as

$$\mathrm{Cone}(\phi)_n = X_n \oplus Y_{n+1} = (X \oplus \Sigma Y)_n$$

with differential

$$d(x, y) = (d_X(x) + (-1)^{n+1}\phi(y), d_Y(y))$$

One of the main properties of this complex is that f is a quasi-isomorphism if and only if $H_*(\mathrm{Cone}(f)) = 0$ (see for example [Weib]).

We consider the complex $\mathrm{Cone}(f)$; it has an obvious filtration induced by the filtration on A, B and C, and

$$\mathrm{grCone}(f) = \mathrm{gr}C \oplus \Sigma[\mathrm{gr}(A\widehat{\otimes}B)] = \mathrm{gr}C \oplus \Sigma[\mathrm{gr}A \otimes \mathrm{gr}B)] = \mathrm{Cone}(\mathrm{gr}f)$$

Since $\mathrm{gr}f$ is a quasi-isomorphism, this complex is acyclic. The lemma above implies that its completion is acyclic, so $\mathrm{Cone}(f)$ is acyclic, namely, f is a quasi-isomorphism. □

3 Hochschild (co)homology: the one variable case

Let us denote $\Psi = \Psi_1$, and $x_\pm = x_{\pm 1}$. Notice that $\widehat{\Psi}^e = \Psi_1\widehat{\otimes}\Psi_1^{op} \cong \Psi_1\widehat{\otimes}\Psi_1 = \Psi_2$. We also use the letters $y_\pm = x_\pm \otimes 1$ and $z_\pm = 1 \otimes x_\pm$ in $\Psi_1 \otimes \Psi_1$. Let W be the 2-dimensional vector space $W = \mathbb{K}e_+ \oplus \mathbb{K}e_-$ The main result of this section is the following:

Proposition 3.1. *The complex*

$$0 \longrightarrow \Psi_2 \otimes \Lambda^2 W \xrightarrow{d_1} \Psi_2 \otimes W \xrightarrow{d_0} \Psi_2 \xrightarrow{m} \Psi_1 \tag{1}$$

is a resolution of Ψ_1 as $\Psi_1\widehat{\otimes}\Psi_1^{op}$-module. The map m is the multiplication, and d_0 and d_1 are determined by the rules

$$d_1(e_+ \wedge e_-) = (y_+ - z_+)e_- - (y_- - z_-)e_+$$

and

$$d_0(e_\pm) = y_\pm - z_\pm \ ,$$

and Ψ_1-linearity on the left and on the right. Explicitly,

$$d_1\Big(\sum_{ijkl} a_{ijkl} y_+^i y_-^j z_+^k z_-^l e_+ \wedge e_-\Big) = \sum_{ijkl} a_{ijkl} y_+^i y_-^j (y_+ - z_+) z_+^k z_-^l e_- - \sum_{ijkl} a_{ijkl} y_+^i y_-^j (y_- - z_-) z_+^k z_-^l e_+$$

Proof. If one declares $|e_\pm| = 1$ and $|e_+ \wedge e_-| = 2$ then this complex is canonically filtered using the Bernstein filtration on Ψ_2 and Ψ_1, and the maps clearly preserve the filtration. The associated graded complex may be identified with the Koszul complex

$$0 \to \mathbb{K}[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \otimes \Lambda^2 W \to \mathbb{K}[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \otimes W \to \mathbb{K}[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \to \mathbb{K}[x_\pm, x_\pm^{-1}] \to 0$$

associated to the regular sequence $\{(y_+ - z_+), (y_- - z_-)\}$ in $\mathbb{K}[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}]$, hence exact. We identify $\mathbb{K}[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}]/\langle y_+ - z_+, y_- - z_-\rangle \cong k[x_\pm, x_\pm^{-1}]$ via the map $z_\pm \mapsto x_\pm$, $y_\pm \mapsto x_\pm$.The proof of the proposition follows from the Lemma above. □

Corollary 3.2. *The algebra $\Psi = \Psi_1$ satisfies a Van den Berg duality property with trivial dualizing module, also called Calabi-Yau property for algebras.*

Proof. One way to prove this corollary is to compute $HH^\bullet(\Psi, \Psi^e)$ using the complex (1), getting Ψ^e in degree $2n$ and zero elsewhere, and then apply Van den Berg's theorem [VdB]; but also we can use this complex to compute homology or cohomology for a general bimodule M, getting the following complexes:

- An homological complex, after applying $M \otimes_{\Psi^e} -$ and identifying $M \otimes_{\Psi^e} \Psi^e \otimes V \cong M \otimes V$:
$$0 \longrightarrow M \otimes \Lambda^2 W \longrightarrow M \otimes W \longrightarrow M \longrightarrow 0$$
The induced differentials are
$$d_1(me_+ \wedge e_-) = (x_+m - mx_+)e_- - (x_-m - mx_-)e_+ = [x_+, m]e_- - [x_-, m]e_+$$
$$d_0(me_+ + m'e_-) = x_+m - mx_+ + x_-m' - m'x_- = [x_+, m] + [x_-, m']$$

- A cohomological complex, after applying $\mathrm{Hom}_{\Psi^e}(-, M)$ and identifying $\mathrm{Hom}_{\Psi^e}(\Psi^e \otimes V, M) \cong V^* \otimes M$:
$$0 \longrightarrow M \longrightarrow W^* \otimes M \longrightarrow \Lambda^2 W^* \otimes M \longrightarrow 0$$
and, in dual bases $e^+ \wedge e^-$, e^+, e^-, the differentials are
$$d^0(m) = [x_+, m]e^+ + [x_-, m]e^-$$
$$d^1(me^+ + m'e^-) = ([x_+, m] - [x_-, m'])e^+ \wedge e^-$$

So, after convenient change of signs and reflecting degrees, we can identify the differentials in homology with cohomology; we conclude that $H^\bullet(\Psi, M) \cong H_{2-\bullet}(\Psi, M)$ for all M. □

Theorem 3.3. *The Hochschild homology and cohomology of Ψ with coefficients in Ψ are given by*
$$HH^0(\Psi) = \mathbb{K},\ HH_2 \cong \Lambda^2 W$$
$$HH^1(\Psi) = \mathbb{K}x_-^{-1}e^+ \oplus \mathbb{K}x_+^{-1}e^- \cong HH_1(\Psi)$$
$$HH^2(\Psi) = \mathbb{K}\frac{1}{x_+x_-}e^+ \wedge e^-,\ HH_0(\Psi) = \mathbb{K}$$

Proof. We know that the center of Ψ is $\mathbb{K}$, this computes HH^0 and, using duality we get $\dim HH_0 = 1$. Also it is well known (and easily computable) that $\Psi/[\Psi, \Psi] = \mathbb{K}x_+^{-1}x_-^{-1}$, so one knows HH_0 and by duality HH^2. It remains to compute HH^1. One can do it directly from the complex (1), but we can also compute using the isomorphism $HH^1(\Psi) = \mathrm{Der}(\Psi)/\mathrm{Innder}(\Psi)$. The second computation will be carried out later; it has the advantage that it gives a standard representative, the "operator" $[\log x_\pm, -]$. □

Lemma 3.4. *The cup product in $H^\bullet(\Psi_1, \Psi_1)$ is non-zero.*

Proof. Let $L_\pm ='' [\log x_\pm, -]''$ be the derivation determined by

$$L_+(x_+) = 0,\ L_+(x_-) = -\frac{1}{x_+}$$

$$L_-(x_-) = 0,\ L_-(x_+) = \frac{1}{x_-}$$

The cup product is defined in the standard complex by the rule $(L_+ \smile L_-)(a \otimes b) = L_+(a)L_-(b)$, as a map $\Psi \otimes \Psi \to \Psi$. for instance

$$(L_+ \smile L_-)(x_- \otimes x_+) = -\frac{1}{x_+}\frac{1}{x_-}$$

$$(L_+ \smile L_-)(x_+ \otimes x_-) = 0$$

Since we work with a smaller complex, in order to compute the product of our cohomology classes we need a comparison map between the small complex and the standard one. In one direction it is not hard, we look at the resolutions:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \Psi \otimes \overline{\Psi}^{\otimes 3} \otimes \Psi & \longrightarrow & \Psi \otimes \overline{\Psi}^{\otimes 2} \otimes \Psi & \longrightarrow & \Psi \otimes \overline{\Psi} \otimes \Psi & \longrightarrow & \Psi \otimes \Psi & \longrightarrow & \Psi \\
\uparrow & & \uparrow & & \uparrow & & \| & & \| \\
0 & \longrightarrow & \Psi \otimes \Lambda^2 W \otimes \Psi & \longrightarrow & \Psi \otimes W \otimes \Psi & \longrightarrow & \Psi \otimes \Psi & \longrightarrow & \Psi
\end{array}$$

and we can show directly that the inclusion $W \to \Psi$ and $\Lambda^2 W \to \Psi \otimes \Psi$ (the second maps $w_1 \wedge w_2 \mapsto w_1 \otimes w_2 - w_2 \otimes w_1$), extend linearly on the left and on the right, giving maps $\Psi \otimes W \otimes \Psi \to \Psi \otimes \Psi \otimes \Psi$ and $\Psi \otimes \Lambda^2 W \otimes \Psi \to \Psi \otimes \Psi^{\otimes 2} \otimes \Psi$, that actually give rise to a complex map. The point is that the difference between $w_1 w_2$ and $w_2 w_1$ is a scalar, so it is zero in $\overline{\Psi}$. As those maps lift the identity, they must be a homotopy equivalence. It is clear that the comparison map gives the corresponding restrictions, namely, if $D : \Psi \to \Psi$ is a derivation (i.e. a cocycle in the standard complex) then it corresponds to the element $D|_W \in \mathrm{Hom}(W, \Psi)$, for example, L_+ corresponds to $-\frac{1}{x_-}e^-$ and L_- to $\frac{1}{x_+}e^+$. We need to show that the class of $L_+ \smile L_- \neq 0$. But

$$(L_+ \smile L_-)(x_+ \otimes x_- - x_- \otimes x_+) = -\frac{1}{x_+}\frac{1}{x_-}$$

so $L_+ \smile L_-$ corresponds to $-\frac{1}{x_+x_-}e^+ \wedge e^-$, which is not zero since it is actually a generator of $H^2(\Psi_1, \Psi_1)$. □

4 Hochschild (co)homology: the n-variable case

In this short section we compute the Hochschild cohomology of Ψ_n using our Künneth formula (Corollary 2.2) and Theorem 3.3 on the one variable case.

Theorem 4.1. *The Hochschild cohomology of Ψ_n with coefficients in Ψ_n is isomorphic to $\Lambda^\bullet D_n$ as graded algebra, where D_n is the $2n$-dimensional vector subspace of $\operatorname{Der}(\Psi_n)$ generated by $[\log x_{i\pm}, -]$. The Gerstenhaber bracket is trivial.*

Proof. First, remark that $\Psi_n = \Psi_{n-1}\widehat{\otimes}\Psi_1$, so the associative algebra structure of $HH^\bullet(\Psi_n)$ is a consequence of the Künneth formula and Lemma 3.4. Once we know that cohomology is generated in degree one, we only check that HH^1 is an abelian Lie algebra. This is a very easy computation, for instance, in one variable, writing $L_\pm := [\log x_\pm, -]$ one can easily check that

$$[L_+, L_-](a) = \left[\sum_{n=1}^{\infty} \frac{x_+^{-n}x_-^{-n}}{(n-1)!^2}, a\right]$$

Namely, this is not zero in $\operatorname{Der}(\Psi_n)$, but it is inner, so it is zero in $H^1(\Psi_n, \Psi_n)$. □

5 The Lie Cohomology of Ψ_n

In this section we prove that $H^1_{Lie}(\Psi_n, \Psi_n)$ (i.e. Lie derivations $\Psi_n \to \Psi_n$ modulo inner derivations) is the 2n+1 dimensional vector space spanned by $[\log x_{\pm i}, -]$ and a particular derivation called f_0. This result is essentially due to Dzhumadil´daev, [D], but our work in previous sections allows us to give a proof which we consider much more transparent than the one appearing in [D]. Lemma 5.1 and 5.2 below are proven in [D] with enough detail.

Lemma 5.1. *Let ϑ denote the element $\vartheta := x_{1+}^{-1}\cdots x_{n+}^{-1}x_{1-}^{-1}\cdots x_{n-}^{-1}$. Then $\Psi_n = [\Psi_n, \Psi_n] \oplus \mathbb{K}\vartheta$. If one denotes f_0 the map $\Psi_n \to \mathbb{K}$ defined by $f_0(\vartheta) = 1$ and $f_0(x_+^I x_-^J) = 0$ for all other monomials, then f_0 is a derivation with respect to the Lie bracket. Observe that f_0 is not a derivation with respect to the associative product.*

Lemma 5.2. *As Lie algebra, $[\Psi_n, \Psi_n]$ is generated by $x_{\pm 1}^{-1}\cdots x_{\pm n}^{-1}$ and $\left\langle x_+^I x_-^J : I, J \in \mathbb{Z}^n_{\geq 0}\right\rangle$.*

Let ϕ be a derivation of Ψ as Lie algebra, then

$$\phi(x_{+1}) = \sum_{nm} a_{nm} x_{+1}^n x_{-1}^m$$

with a_{nm} formal series on the variables $x_{2\pm}, \ldots, x_{n\pm}$. Since

$$[\sum_{nm} b_{nm} x_{+1}^2 x_{-1}^m, x_{+1}] = \sum_{nm} m b_{nm} x_{+1}^n x_{-1}^{m-1}$$

then, except for the term corresponding to x_{-1}^{-1}, all other terms (of $\phi(x_{+1})$) can be written as the bracket of an element with x_{+1}, hence, modulo inner derivation, we may assume that

$$\phi(x_{+1}) = \sum_{nm} a_n x_{+1}^n x_{-1}^{-1}$$

Using the relation

$$[x_{-1}, x_{+1}] = 1$$

and that every derivation preserves the center, we have

$$[\phi(x_{-1}), x_{+1}] + [x_{-1}, \phi(x_{+1})] = D(1) = c \in k$$

If one writes

$$\phi(x_{-1}) = \sum_{nm} c_{nm} x_{+1}^n x_{-1}^m$$

then

$$\sum_{nm} c_{nm} m x_{+1}^n x_{-1}^{m-1} + \sum_n a_n n x_{+1}^{n-1} x_{-1}^{-1} = c \in k$$

This equality implies two facts: first, every term with $m \neq 0$ must be zero, and so $\phi(x_{-1}) = \sum_n c_n x_{+1}^n$, and secondly, if $n \neq 0$ then also $a_n = 0$, that is $\phi(x_{+1}) = a x_{-1}^{-1}$.

Notice that if we change ϕ by $\widetilde{\phi} := \phi + [\sum_n d_n x_{+1}^n, -]$, (with d_n formal series in the other variables) then $\widetilde{\phi}$ takes the same value at x_{+1}, but changes the value of $\phi(x_{-1})$ by adding $[\sum_n d_n x_{+1}^n, x_{-1}] = \sum_n n d_n x_{+1}^{n-1}$, so we see that we can choose d_n in such a way that this new value is zero, except eventually the term with power -1. That is, modulo inner derivation we have

$$\phi(x_{+1}) = a_+ x_{-1}^{-1},\ \phi(x_{-1}) = a_- x_{+1}^{-1}$$

with $a_\pm$ series in variables different from $x_{\pm 1}$. As immediate consequence of this and the relation $[x_{-1}, x_{+1}] = 1$ is that $\phi(1) = 0$. Another consequence is, assuming that ϕ take those values, taking $i \neq 1$ and the other relations $[x_{\pm 1}, x_{\pm i}] = 0$, we have

$$[\phi(x_{\pm 1}), x_{\pm i}] + [x_{\pm 1}, \phi(x_{\pm i})] = 0$$

Let us write

$$\phi(x_{\pm i}) = \sum_{nm} b_{\pm,nm} x_{+i}^n x_{-i}^m$$

where the b's are series in the variables different from i. We get

$$0 = [a_\pm x_{\mp 1}^{-1}, x_{\pm i}] + [x_{\pm 1}, \phi(x_{\pm i})] = [a_\pm, x_{\pm i}] x_{\mp 1}^{-1} + \sum_{nm} [x_{\pm 1}, b_{\pm,n,m}] x_{+i}^n x_{-i}^m$$

But for any series b, writing $b = \sum_n b_n x_{+1}^n$ with b_n independent of x_{+1}, we have as before $[\sum_n b_n x_{+1}^n, x_{-1}] = \sum_n n b_n x_{+1}^{n-1}$ so there is no term with x_{+1}^{-1}; we conclude that $[x_{\pm 1}, b]$ can never be of the form $a x_{\mp}^{-1}$ (with a a series independent of $x_{\pm 1}$). This implies two things: first

$$[a_\pm, x_{\pm i}] = 0$$

namely, that $a_\pm$ are constants, and second

$$[b_{\pm,nm}, x_{\pm 1}] = 0$$

that is, the $b_{\pm,n,m}$'s -and consequently $\phi(x_{\pm i})$- do not depend on $x_{\pm 1}$. With this result we argue by induction and we conclude the following statement

Lemma 5.3. *Modulo inner derivations, all Lie derivations take the values*

$$\phi(x_{\pm i}) = a_{\pm i} x_{\mp i}^{-1}$$

with $a_{\pm i} \in k$.

Corollary 5.4. *Modulo inner derivations and the set of derivations* $\{[\log x_{i\pm}, -]\}_i$*, every Lie derivation vanishes on* $x_{i\pm}$.

Remark 5.5. The assignment

$$\begin{cases} x_+ & \mapsto & +x_- \\ x_- & \mapsto & -x_+ \end{cases}$$

induce a well defined isomorphism of associative algebra $\mathcal{F} : \Psi_1 \to \Psi_1$ that one can call the Fourier transform. Even thought it is not strict necessary to introduce this transform for the proof of next result, it helps simplifying the arguments.

Corollary 5.6. $HH^1(\Psi_n)$ *is the* $2n$ *dimensional vector space generated by* $\{[\log x_{i\pm}, -]\}_i$.

Proof. If D is an associative derivation, then in particular it is a Lie derivation, and the Lie-inner derivations are also associative derivations, so modulo $\{[\log x_{i\pm}, -]\}_i$ (notice that these are associative derivations) we have that D vanishes on the $x_{i\pm}$. But if D is an associative derivation and vanishes on the $x_{i\pm}$ then $D = 0$. We conclude that the log's generate $HH^1(\Psi_n)$. We need to see that they are linearly independent. But considering $\mathcal{F}_i$ the Fourier transform with respect to the variable x_i, we have that $D_{\pm j} := \log x_{+j} \pm \log x_{-j}$ are all eigenvectors of the F_i, with different eigenvalues, so they are l.i. □

Remark 5.7. This corollary finishes the proof of Theorem 4.1.

Lemma 5.8. *Let* $D : \Psi_n \to \Psi_n$ *be a derivation with respect to the Lie bracket, if* $D(x_{\pm i}) = 0$ *(and* D *is continuous with respect to the filtration) then* D *is a scalar multiple of* f_0.

Proof. First recall $D(1) = D[x_{+i}, x_{-i}] = [Dx_{+i}, x_{-i}] + [x_{+i}, Dx_{-i}] = 0$. Also, for $i \neq j$

$$0 = [x_{\pm j}, x_{+i}^n] \Rightarrow [x_{\pm j}, Dx_{+i}^n] = 0$$

that is, $D(x_{+i}^n)$ only depends on $x_{\pm i}$, but using $0 = [x_i, x_{+i}^n]$ we get that $D(x_{+i}^n)$ only depends on x_{-i}. Also

$$[x_{-i}, x_{+i}^n] = nx^{n-1} \Rightarrow [x_{-i}, D(x_{+i}^n)] = nD(x_{+i}^{n-1})$$

but because $D(x_{+i}^n)$ only depends on x_{-i}, it follows that the bracket on the left is zero. We conclude $D(x_{+i}^n) = 0$ for $n \neq -1$. By a symmetry argument (using Fourier transform) we conclude the same for x_{-i}^n.

Also, for $j \neq i$, $x_{\pm i}^{-1}$ commutes with $x_{\pm j}$, and so $D(x_{\pm i}^{-1})$ depends only on the variables $x_{\pm i}$. Using that x_{+i}^{-1} commutes with x_i, we have that $D(x_{+i}^{-1})$ commutes with x_{+i}, that is, depends on x_{-i}, but from

$$[x_{-i}, x_{+i}^{-1}] = -x_{+i}^{-2}$$

and the fact that $D(x_{+i}^{-2}) = 0$ it follows that $D(x_{+i}^{-1})$ is constant. Let us denote $E_i := x_{+i}x_{-i}$, we know $D(E_i) = 0$ and $[E_i, x_{+i}^{-1}] = -x_{+i}^{-1}$, so

$$0 = [E_i, const] = -const$$

hence $D(x_{+i}^{-1}) = 0$. By Fourier considerations we also have $D(x_{-i}^{-1}) = 0$.

For an arbitrary differential operator, we argue by induction on the sum of the total degrees, namely, we consider a monomial of type

$$w = x_{+1}^{n_1} x_{-1}^{m_1} x_{+2}^{n_2} x_{-2}^{m_2} \cdots x_{+k}^{n_k} x_{-k}^{m_k}$$

(with m_i and n_i non negatives). We know that $[w, x_{\pm i}]$ is a monomial with total degree strictly smaller, and so, by inductive argument $D[w, x_{\pm i}] = 0$, and also it is equal to $[Dw, x_{\pm i}]$, so we conclude that $D(w)$ is constant. Using

$$[x_+^n x_-, x_+ x_-^m] = (1 - mn)x_+^n x_-^m - \frac{1}{2}mn(n-1)(m-1)x_+^{n-1}x_+^{m-1} + \cdots$$

We compute (assume $n_1 > 1$)

$$[x_1^{n_1-1}x_{-1}, x_{+1}x_{-1}^{m_1}x_{+2}^{n_2}x_{-2}^{m_2} \cdots x_{+k}^{n_k}x_{-k}^{m_k}] = (1 - mn)x_+^n x_-^m x_{+2}^{n_2}x_{-2}^{m_2} \cdots x_{+k}^{n_k}x_{-k}^{m_k}$$
$$-\frac{1}{2}mn(n-1)(m-1)x_+^{n-1}x_+^{m-1}x_{+2}^{n_2}x_{-2}^{m_2} \cdots x_{+k}^{n_k}x_{-k}^{m_k} + \cdots$$

Because of depending on one variable, or because of having smaller exponent, we conclude that D vanishes on monomials corresponding to differential operators of nonnegative total degree, hence, D vanish on A_n.

Now we know D vanishes on A_n and $x_{i\pm}^{-1}$, using 5.2 we conclude that D vanishes on $[\Psi_n, \Psi_n]$.

□

Now we can conclude the following statement:

Theorem 5.9. *The $HH^1(\Psi_n, \Psi_n)$ is $2n$-dimensional generated by $\log x_{i+}, \log x_{i-}$ for $i = 1, ..., n$. Moreover, $H^1_{Lie}(\Psi_n, \Psi_n)$ is $(2n+1)$-dimensional generated by $\log x_{i+}, \log x_{i-}$ $(i = 1, ..., n)$ and f_0.*

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